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The identification of the piecewise homogeneous thermal conductivity of conductors subjected to a heat flow test

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Abstract

The identification of the piecewise homogeneous thermal conductivity of conductors with unknown location of the discontinuities is investigated using additional boundary and/or interior measurements of the temperature in heat-flow experiments. A mathematical analysis is performed for the steady-state case, whilst for the unsteady case a numerical method based on the boundary element method, as a direct solution procedure, combined with an ordinary least-squares technique subject to bounds on the variables, is employed for obtaining an inverse numerical solution. The sensitivity coefficients for the unspecified boundary conditions and for interior temperatures are calculated and show the need for interior measurement information to be imposed. For a heat conductor presenting a single discontinuity, it was found that, when the number of time measurements is limited then two interior temperature measurements are necessary and sufficient in order to render a good estimate of the exact solution, otherwise one may increase the number of time measurements at a single interior sensor location to render the same result.

Key words: Inverse coefficient identification problem; Discontinuous thermal conductivity; Boundary element method (BEM), Sensitivity coefficients

1. Introduction

The aim of this paper is to investigate the identification of the piecewise homogeneous thermal conductivity of conductors with unknown location of the discontinuities using additional boundary and/or interior measurements of the temperature in transient heat conduction experiments. This formulation models the presence of 'faults' within conductors which are defined herein as discontinuity points for the thermal conductivity of the conductors. Practically, this formulation models a heat conduction experiment in which two or more homogeneous conductors are butted together and the experiment performed over the whole of the linked material.

The case of unknown fault location, but with two known conductivity constant coefficients, has been investigated analytically in degenerate hypotheses by Cannon [1]. The case of unknown piecewise homogeneous conductivity coefficient, but with known fault location, has been briefly tested numerically by Carrera and Neuman [2]. It appears that the case of the simultaneous determination of the piecewise homogeneity of the thermal conductivity and the location of the unknown discontinuity is not encountered in the literature and therefore this study is aimed at investigating such a situation. Here we develop solutions to the case where a constant-rate of heat flow is applied to a sample which has a step change in the conductivity at an unknown position along its length, due, for example to a presence of a modification of the type of material representing the conductor. The lack of knowledge of these quantities is compensated for by measuring in time boundary and/or internal values of the temperature using thermocouples penetrating the material.

A simple one-dimensional, transient mathematical formulation of the problem under investigation is introduced in Section 2 for a temperature build-up and decay,

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transient, heat conduction experiment. Then the steady-state mathematical analysis is performed in Section 3 in order to gain insight into the uniqueness features of the inverse problem and to argue the need for investigating the transient situation using the numerical methods and analysis which are developed in the subsequent sections.

In Sections 4 and 5 the boundary element method (BEM) is combined with an ordinary, nonlinear least-squares method subject to physically sound constraints, for obtaining an inverse numerical solution. The BEM does not require any domain discretisation and therefore it reduces by one the degree of the dimensionality of the problem. Consequently, the BEM does not require any moving meshes, needed when using iteratively domain discretisation methods, e.g. finite-difference method (FDM) or the finite element method (FEM), for the determination of the spatial fault location. Furthermore, unlike the FDM and FEM, there is no need of interpolation onto grid cells when using the BEM for imposing the measured internal temperature values.

Finally, in Section 6 the accuracy of the BEM as a direct problem solution procedure is compared with that obtained using a FDM. Also, a sensitivity coefficient analysis shows, prior to inversion, the need for internal temperature measurements. The effects on the uniqueness of the numerical solution of the number and location of the sensors recording time measurements of the temperature are thoroughly investigated. The numerical results are discussed and compared with their exact values for numerically simulated input data, both with and without noise included.

2. Mathematical formulation

The governing heat conduction equation is given by

$$C\frac{\partial T}{\partial t}(x,t) = \frac{\partial}{\partial x}\left(k(x)\frac{\partial T}{\partial x}(x,t)\right), \quad 0 < x < L, \quad 0 < t < \infty \qquad (1)$$

where T is the temperature, C is the volumetric heat capacity which is assumed to be constant, $k(x)$ is the thermal conductivity and L is the length of the heat conductor.

For composite homogeneous conductors the thermal conductivity is piecewise constant, i.e.

$$k(x) = k_i \quad \text{for } x_{i-1}^s \leqslant x \leqslant x_i^s, \quad i = \overline{1, N_s} \qquad (2)$$

where N_s is the number of zonations (layers) of homogeneous materials butted together, $x_0^s = 0$, $x_{N_s}^s = L$ and $x_i^s \in (0, L)$ for $i = \overline{1,(N_s-1)}$, are the unknown discontinuity locations.

The boundary conditions are derived for a heat flow test similar to those of Garnier et al. [3] and Dowding et al. [4] and is shown schematically in Fig. 1. In this experiment an inflow of heat is applied using a heater at the entrance of an upstream block of material which is in thermal contact with the heat conductor sample and the induced temperature is measured with a differential transducer connected between the two extreme faces of the sample. After the transients have past, or even before that, the heater may be switched off.

At the start of the experiment, the temperature of the sample is constant and is set zero, i.e.

$$T(x,0) = 0 \quad \text{for } 0 \leqslant x \leqslant L. \qquad (3)$$

The exit face of the sample, $x = L$, is buffered at a constant temperature, set to zero, i.e.

$$T(L,t) = 0 \quad \text{for } 0 < t < \infty. \qquad (4)$$

At the entrance face of the sample, $x = 0$, conservation of heat at the sample–block interface is applied, resulting in a boundary condition of the fourth-kind, namely,

$$S_u\frac{\partial T}{\partial t}(0,t) = k(0)A\frac{\partial T}{\partial x}(0,t) + Q_u, \quad 0 < t < \infty \qquad (5)$$

where $S_u = C_u V_u$ is the heat capacity of the upstream block of material with volume V_u and volumetric heat capacity C_u, A is the cross-sectional area of the sample and Q_u is the constant inflow rate of heat at the entrance of the upstream block which is zero when the heater is off.

At the interfaces $x = x_i^s$ for $i = \overline{1,(N_s-1)}$, the continuity of the temperature and the heat flux are applied, namely,

$$\lim_{x \nearrow x_i^s} T(x,t) = \lim_{x \searrow x_i^s} T(x,t), \quad 0 < t < \infty \qquad (6)$$

$$\lim_{x \nearrow x_i^s} k_i \frac{\partial T}{\partial x} T(x,t) = \lim_{x \searrow x_i^s} k_{i+1}\frac{\partial T}{\partial x}(x,t), \quad 0 < t < \infty. \qquad (7)$$

The inverse problem requires, in addition to the temperature solution $T(x,t)$, the determination of the piecewise homogeneous values k_i for $i = \overline{1, N_s}$, and the unknown fault locations x_i^s for $i = \overline{1,(N_s-1)}$. The lack of information is compensated for by additional temperature measurements at N_w sensor locations x_i within the medium $[0, L)$, recorded in time at N_T prescribed instants t'_j

$$T(x_i, t'_j) = [T(x_i, t'_j)]^{(e)} = \text{measured}, \quad i = \overline{1, N_w}, \quad j = \overline{1, N_T}. \qquad (8)$$

3. Steady-state analysis

The steady-state solution of eqns (1)–(7) is given by

$$T(x,\infty) = \frac{Q_u}{A}\left[-\frac{x}{k_i} + \frac{L}{k_{N_s}} + \sum_{j=i}^{N_s-1} x_j^s\left(\frac{1}{k_j} - \frac{1}{k_{j+1}}\right)\right], \quad x_{i-1}^s \leqslant x \leqslant x_i^s, \quad i = \overline{1, N_s} \qquad (9)$$

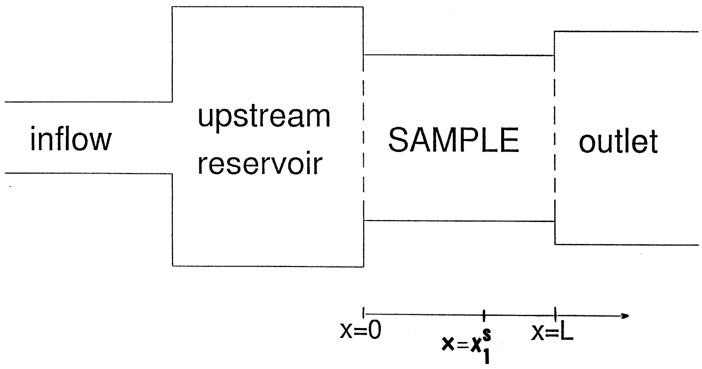

Fig. 1. Schematic diagram showing the experiment arrangement of a heat flow test.

from which it can be seen that the boundary measurement of the temperature at $x = 0$, say $T(0, \infty) = T_\infty$, and one and only one measurement of the steady-state temperature at a single location y_i within each layer (x^s_{i-1}, x^s_i), say $T(y_i, \infty) = T_i$ for $i = \overline{1, N_s}$, can be used to provided $(N_s + 1)$ possibly linearly independent equations. However, the number of unknowns in equation (9) is $(2N_s - 1)$ and hence the steady-state mathematical analysis can be solely used only if $N_s + 1 > 2N_s - 1$, or $N_s \in \{1, 2\}$. If $N_s = 1$ then

$$k_1 = \frac{Q_u y_1}{A(T_\infty - T_1)}, \quad L = \frac{T_\infty y_1}{T_\infty - T_1},$$
$$T(x, \infty) = T_\infty - \frac{x(T_\infty - T_1)}{y_1} \tag{10}$$

whilst, if $N_s = 2$ then

$$k_1 = \frac{Q_u y_1}{A(T_\infty - T_1)}, \quad k_2 = \frac{Q_u(L - y_2)}{AT_2},$$
$$x^s_1 = \frac{(T_\infty - T_2)Ly_1 - T_2 T_\infty y_1}{(T_\infty - T_1)(L - y_2) - T_2 y_1} \tag{11}$$

$$T(x, \infty) = \begin{cases} \dfrac{1}{y_1(L - y_2)}[T_2 y_1(L - x^s_1) + (T_\infty - T_1)(L - T_2)(x^s_1 - x)], & 0 \leqslant x \leqslant x^s_1 \\ \dfrac{(L - x)T_2}{L - y_2}, & x^s_1 \leqslant x \leqslant L. \end{cases} \tag{12}$$

Then, we can retrieve uniquely both the conductivity and the length of a homogeneous conductor from only one boundary, at $x = 0$, and one internal, $x = y_1$, steady-state measurement of the temperature, whilst as given by equations (11) and (12) we can retrieve uniquely both the two conductivities and the location of the discontinuity from only one boundary, at $x = 0$, and two internal, $x = = \{y_1, y_2\}$, static temperature measurements. However, there are possible drawbacks when using the steady-state approach, namely:

(i) the steady-state may not exist, or long times, which are practically expensive, may be required to achieve the steady-state;
(ii) only one or no internal measurement of the temperature may be available;
(iii) the transient temperature distribution is also requested;
(iv) $N_s > 2$.

Therefore, in order to deal with such possible practical difficulties the transient case needs to be considered. For $N_s = 1$, an analytical solution has recently been given by Esaki et al. [5], but for composite conductors, i.e. $N_s > 1$, there is as yet no analytical study available and therefore both the direct and inverse problems (1)–(7) are investigated numerically as described in the next sections.

4. Boundary element solution of the direct problem

The first step in the inverse analysis which will be finally undertaken is the development of the corresponding direct solution for the problem, i.e. equation (1)–(7), when k_i for $i = \overline{1, N_s}$, and x^s_i for $i = \overline{1, (N_s - 1)}$, are assumed to be known.

In this study, the numerical method adopted for solving the direct initial boundary value problem is the BEM. If,

$$\alpha(x) = \alpha_i = k_i/C \quad \text{for } x^s_{i-1} \leqslant x \leqslant x^s_i, \quad i = \overline{1, N_s} \tag{13}$$

is defined to be the piecewise constant thermal diffusivity of the medium, then the governing heat conduction equation (1), in which the thermal diffusivity is given by equation (13) over each zonation $[x^s_{i-1}, x^s_i]$, possesses a fundamental solution, namely,

$$FS_i(x, t; \xi, \tau) = \frac{H(t - \tau)}{(4\pi\alpha_i(t - \tau))^{1/2}} \exp\left(-\frac{(x - \xi)^2}{4\alpha_i(t - \tau)}\right),$$
$$x^s_{i-1} \leqslant x, \xi \leqslant x^s_i, \quad t, \tau > 0, \quad i = \overline{1, N_s} \tag{14}$$

where H is the Heaviside function. The use of the fundamental solution given by equation (14) enables us to reformulate the partial differential equation (1) over each zonation $[x^s_{i-1}, x^s_i]$ as a boundary integral equation, as follows:

$$\eta(x)T(x,t) = \int_0^{t_f} \alpha_i T'(x^s_{i-1},\tau) FS_i(x,t;x^s_{i-1},\tau)\,\mathrm{d}\tau$$

$$+\int_0^{t_f} \alpha_i T'(x^s_i,\tau) FS_i(x,t;x^s_i,\tau)\,\mathrm{d}\tau$$

$$-\int_0^{t_f} \alpha_i T(x^s_{i-1},\tau) FS'_i(x,t;x^s_{i-1},\tau)\,\mathrm{d}\tau$$

$$-\int_0^{t_f} \alpha_i T(x^s_i,\tau) FS'_i(x,t;x^s_i,\tau)\,\mathrm{d}\tau$$

$$+\int_{x^s_{i-1}}^{x^s_i} T_0(y) FS_i(x,t;y,0)\,\mathrm{d}y, \quad (x,t)\in[x^s_{i-1},x^s_i]\times(0,t_f] \tag{15}$$

where t_f is a final time of interest over which duration the experiment is performed, $\eta(x)$ is a coefficient function equal to unity if $x\in(x^s_{i-1},x^s_i)$ and 0.5 if $x\in\{x^s_{i-1},x^s_i\}$, the prime denotes the differentiation with respect to the outward normal n at the boundaries $x\in\{x^s_{i-1},x^s_i\}$ and $T_0(y)$ is the initial temperature which is zero, see equation (3), if the heater is on and which is equal to $T(y,t_f)$, (obtained by first solving the heater-on flow test), when the heater is switched off. Therefore, when the heater is on, the BEM requires the discretisation of the boundary space solution domain, $x\in\{0,1\}$, only, and it fully reduces the dimensionality of the problem by one. When the heater is off the BEM requires an additional solution space domain discretisation at $t=t_f$, but the line integral involved in equation (15) can easily be evaluated numerically without much computational cost.

The BEM discretisation of the boundary integral equation (15) is performed by subdividing the time interval $[0,t_f]$ into N equal time intervals $[t_{j-1},t_j]$, for $j=\overline{1,N}$, and, for simplicity, we assume that the boundary temperature and its normal derivative are constant over each time step and take their values at the midpoint, $\bar{t}_j=(t_{j-1}+t_j)/2$, namely,

$$T(x^s_i,t)^{\mp} \simeq T(x^s_i,\bar{t}_j)^{\mp} = (T^i_j)^{\mp},$$
$$T'(x^s_i,t)^{\mp} \simeq T'(x^s_i,\bar{t}_j)^{\mp} = (T'^i_j)^{\mp},$$
$$t\in[t_{j-1},t_j] \tag{16}$$

where $\mp$ denotes the limits of the functions involved from the left and right at the interface $x=x^s_i$ for $i=\overline{1,(N_s-1)}$, respectively. When the heater is switched off we also need the discretisation of the space intervals $[x^s_{i-1},x^s_i]$ into N^i_0 cells, namely, $[y^i_{k-1},y^i_k]$, for $k=\overline{1,N^i_0}$, $i=\overline{1,N_s}$, and this is achieved using a constant space cell approximation at $y=\bar{y}^i_k=(y^i_{k-1}+y^i_k)/2$, namely,

$$T_0(y) = T(y,t_f) \simeq T(\bar{y}^i_k,t_f) = T^i_{0k},$$
$$y\in[y^i_{k-1},y^i_k], \quad k=\overline{1,N^i_0}, \quad i=\overline{1,N_s}. \tag{17}$$

Based on the constant BEM approximations as given by equations (16) and (17), equation (15) becomes

$$\eta(x)T(x,t) = \sum_{j=1}^{N}\left[\alpha_i T'^{i-1}_j \int_{t_{j-1}}^{t_j} FS_i(x,t;x^s_{i-1},\tau)\,\mathrm{d}\tau\right.$$

$$+\alpha_i T'^i_j \int_{t_{j-1}}^{t_j} FS_i(x,t;x^s_i,\tau)\,\mathrm{d}\tau$$

$$-\alpha_i T^{i-1}_j \int_{t_{j-1}}^{t_j} FS'_i(x,t;x^s_{i-1},\tau)\,\mathrm{d}\tau$$

$$\left.-\alpha_i T^i_j \int_{t_{j-1}}^{t_j} FS'_i(x,t;x^s_i,\tau)\right]\mathrm{d}\tau$$

$$+\sum_{k=1}^{N^i_0} T^i_{0k} \int_{y^i_{k-1}}^{y^i_k} FS_i(x,t;y,0)\,\mathrm{d}y,$$

$$(x,t)\in[x^s_{i-1},x^s_i]\times(0,t_f], \quad i=\overline{1,N_s} \tag{18}$$

where, for simplicity, the symbol $\mp$ has been dropped. Letting x in equation (18) tend to the interface locations x^s_i for $i=\overline{0,N_s}$, we obtain a system of linear equations involving boundary and interface unknowns only. This system of equations is completed with the boundary conditions (4)–(7), which in discretised form yield,

$$T^{N_s}_j = 0, \quad j=\overline{1,N} \tag{19}$$

$$S_u\left(\frac{T^0_1 - T_0(0)}{t_f/(N)}\right) + k_1 A T'^0_1 = Q_u \tag{20}$$

$$S_u\left(\frac{T^0_{j+1} - T^0_{j-1}}{2t_f/N}\right) + k_1 A T'^0_j = Q_u, \quad j=\overline{2,(N-1)} \tag{21}$$

$$S_u\left(\frac{T^0_N - T^0_{N-1}}{t_f/N}\right) + k_1 A T'^0_N = Q_u \tag{22}$$

$$(T^i_j)^- = (T^i_j)^+,$$
$$k_i(T'^i_j)^- = -k_{i+1}(T'^i_j)^+,$$
$$i=\overline{1,(N_s-1)} \tag{23}$$

where $T_0(0)=0$ if the heater is on and $T_0(0)=T^0_N$ (an already calculated value) if the heater is off.

5. The least-squares minimization technique

In the inverse problem given by equations (1)–(7) in which $\underline{k}=(k_i)$ for $i=\overline{1,N_s}$, and $\underline{x}^s=(x^s_i)$ for $i=\overline{1,(N_s-1)}$, are unknown, additional temperature measurements are imposed. We can then compare the measured, (e), and the computed, (c), data in equation (8) in the least-squares sense

$$LS(\underline{k}, \underline{x}^s) = \sum_{i=1}^{N_w} \sum_{j=1}^{N_T} \lambda_i |T(x_i, t_j'; \underline{k}, \underline{x}^s)^{(e)} - T(x_i, t_j'; \underline{k}, \underline{x}^s)^{(c)}|^2 \quad (24)$$

where the parameters λ_i are set to be 0 or 1 according to whether we measure in time at the space locations, $x = x_i$, the multiplying quantities, i.e. $T(x_i, t_j'; \underline{k}, \underline{x}^s)^{(e)}$, or not. The least-squares functional (24), is minimized using the NAG routine E04UCF over the set

$$M = \{(\underline{k}, \underline{x}^s) \mid \underline{bu}^k > \underline{k} > \underline{bl}^k, \underline{bu}^s > \underline{x}^s > \underline{bl}^s, k_i \neq k_j, \forall i \neq j\} \quad (25)$$

where $\underline{bu}^k$, $\underline{bu}^s$, $\underline{bl}^k$ and $\underline{bl}^s$ are vectors of upper and lower bounds, respectively, to which the physical variables are subjected.

6. Numerical results and discussion

The numerical analysis performed in the previous sections has been tested for a composite heat conductor formed from silver butted together with aluminum, so presenting a single fault, i.e. $N_s = 2$, whilst the case of multiple faults situation is deferred to a future study. This type of composite material has been investigated since silver and aluminum have approximately the same volumetric heat capacity $C = 2.46$ J cm^{-3} °C^{-1}. The whole linked material has a length of $L = 1.16$ cm and a cross-sectional area of $A = 2.75$ cm^2. An inflow rate of heat of $Q_u = 10^4$ W is applied for $t_f \approx 33.7$ min at the entrance of an upstream block of material with heat capacity $S_u = 10^3$ W min °C^{-1}. The single fault is located at $x_1^s = (20L/35) \approx 0.662$ cm and the piecewise constant thermal conductivity is given by

$$k(x) = \begin{cases} k_1 = 4.190 \text{ W cm}^{-1}\ ^\circ\text{C}^{-1}, & 0 \leqslant x < x_1^s \\ k_2 = 2.095 \text{ W cm}^{-1}\ ^\circ\text{C}^{-1}, & x_1^s < x \leqslant L \end{cases}. \quad (26)$$

The BEM discretisation of the problem under investigation has been performed using $N = 45$ time elements and, when the heater is off, $N_0^1 = N_0^2 = 6$ cells.

For the direct problem, Fig. 2 shows the numerical BEM temperature build-up and decay curves, as a function of time, at various space locations within the medium, or on the boundary $x = 0$. From Fig. 2 it can be seen that the maximum temperature values are obtained at the active thermal boundary $x = 0$ and the temperature decays monotonically to zero as we approach the boundary $x = L$ where a zero temperature condition, as given by equation (4), is prescribed. Except for the temperature curves at the unknown fault location $x = x_1^s = (20L/35)$, the temperature curves from Fig. 2 calculated at some particular prescribed instants t_j' for $j = \overline{1, N_T}$, will be considered as exact input data, as given by equation (8), in the inverse analysis.

As no analytical solution is available for the problem under investigation the accuracy of the BEM utilised for solving the direct problem, as described in Section 4, is compared with a weighted average Galerkin finite-difference method (FDM), similar to that used by Flach and Ozisik [6] and Huang and Ozisik [7], but with continuous spacewise varying thermal properties and Neumann boundary conditions. Since the lines of constant temperature obtained throughout the medium were indistinguishable for the two numerical approaches, it can be argued that the FDM and the BEM perform equally well as numerical solutions for the direct problem.

Prior to investigating the inverse problem, it is useful to calculate the sensitivity coefficients which are the first derivatives of the temperature at sensor measurement locations x_i for $i = \overline{1, N_w}$, with respect to the unknowns, namely, k_i for $i = \overline{1, N_s}$, and x_i^s for $i = \overline{1, (N_s - 1)}$, as a function of time. They provide indicators of how well-designed is the experiment and, in general, the sensitivity coefficients are desired to be large and uncorrelated, i.e. linearly independent, see Beck and Arnold [8]. A sense of the magnitude of the sensitivity coefficients is gained through normalisation by multiplying them with their corresponding unknown differentiation variable, resulting in units of temperature for the normalised sensitivity coefficients. The degree of uncorrelation of these coefficients can then be illustrated by the departure of their ratios from a constant value. Based on this criterion we can determine the optimal space locations and time instant measurements to be imposed or recorded in equation (8).

For $N_s = 2$, by calculating the ratios between the sensitivity coefficients, namely

$$R_1(x, t) = x_1^s \frac{\partial T}{\partial x_1^s} \Big/ k_1 \frac{\partial T}{\partial k_1},$$
$$R_2(x, t) = k_1 \frac{\partial T}{\partial k_1} \Big/ k_2 \frac{\partial T}{\partial k_2}, \quad R_3(x, t) = k_2 \frac{\partial T}{\partial k_2} \Big/ x_1^s \frac{\partial T}{\partial x_1^s} \quad (27)$$

it was observed that the sensitivity coefficients at $x = 0$ possess two degrees of correlation, namely,

$$R_1(0, t) \approx 1; \quad R_2(0, t) \approx \tfrac{2}{3}; \quad R_3(0, t) \approx \tfrac{3}{2}. \quad (28)$$

Therefore, we may conclude that only one of the three unknowns x_1^s, k_1 or k_2, assuming the other two are fixed and known, can be identified from only temperature measurements imposed in equation (8), at the boundary $x = 0$. Based on this sensitivity analysis, which corresponds to information supplied only by boundary temperature measurements, it is concluded that interior measurements are necessary in order to reduce the non-uniqueness of the ill-posed problem.

The ratios $R_i(x, t)$ for $i \in \{1, 2, 3\}$, between the sensitivity coefficients at the interior locations $x = (x_1^s/2)$ and $x = (3x_1^s/2)$ showed single degrees of correlations, namely,

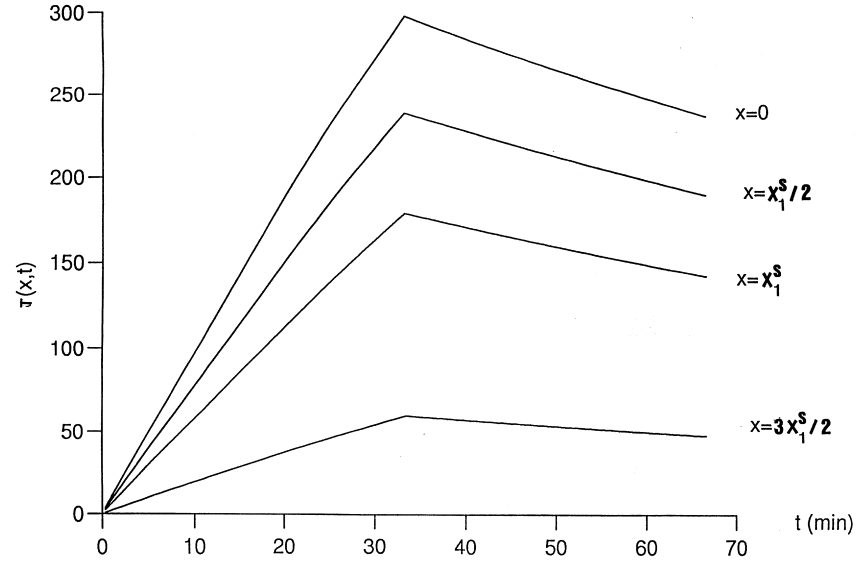

Fig. 2. The temperature $T(x, t)$ as a function of time, at the space locations, $x \in \{0, x_1^s/2, x_1^s, 3x_1^s/2\} = \{0, 10L/35, 20L/35, 30L/35\} \approx \{0, 0.331, 0.662, 0.994\}$cm, obtained using the BEM for the direct problem.

$$R_3\left(\frac{x_1^s}{2}, t\right) \approx \frac{3}{2}; \quad R_1\left(\frac{3x_1^s}{2}, t\right) \approx 1. \tag{29}$$

Therefore, we conclude that we cannot identify simultaneously both x_1^s and k_2 from only temperature measurements imposed in equation (8) at $x = (x_1^s/2)$, whilst from only temperature measurements imposed in equation (8) at $x = (3x_1^s/2)$ we cannot identify simultaneously both x_1^s and k_1. Therefore, in such situations, in general, one needs to fix one of the unknowns.

So far, the sensitivity analysis performed in this subsection highlights the highly ill-posed non-unique solution problem that we have formulated when the number of spatial observations assumed available is restricted to $N_w = 1$. Finally, it should be mentioned that the same sensitivity analysis has also been performed for the boundary heat flux but the same two degrees of correlation between the corresponding flux sensitivity coefficients, similar to those given by equation (28), have been obtained. In addition, the sensitivity coefficients for the boundary heat flux are small, of $\mathbf{O}$ (10^{-3}) W cm^{-2}, and therefore they are not presented in this study.

In the inverse analysis of solving equations (1)–(8) for a material containing a single discontinuity, i.e. $N_s = 2$, the number of coefficient unknowns is equal to $(2N_s - 1) = 3$, and in order to investigate the most limited time recording information, the number of time measurements at each sensor location was initially taken to be $N_T = 3$. The time measurements t'_j at each of the locations $x_i \in \{0, x_1^s/2, 3x_1^s/2\} = \{0, 0.331, 0.994\}$ cm, were selected as

$$(t'_1, t'_2, t'_3) = \begin{cases} (\bar{t}_{10}, \bar{t}_{21}, \bar{t}_{55}) = (7.116, 15.356, 40.449) \text{ min}, & x = 0 \\ (\bar{t}_5, \bar{t}_{10}, \bar{t}_{15}) = (3.371, 7.116, 10.861) \text{ min}, & x = \dfrac{x_1^s}{2} \\ (\bar{t}_7, \bar{t}_{16}, \bar{t}_{25}) = (4.869, 11.610, 18.352) \text{ min}, & x = \dfrac{3x_1^s}{2} \end{cases}. \tag{30}$$

It should be noted that this selection of the time measurements can be regarded as optimal mainly for exact or errors in the data of the same order as the computer machine precision, with the choice based on investigating closely the approximative sign $\approx$ in equation (29) which gives the largest departure of the ratios of the sensitivity coefficients from a constant function. When realistic noisy data is considered, and the number of spatial locations is limited to $N_w = 1$, there is no optimal decision with respect to a possible meaningful criterion to be made and the earlier conclusions on non-identifiability of the ill-posed problem when $N_w = 1$ apply. However, it may be that when gathering the information at the time measurements given by equation (30), when $N_w \in \{2, 3\}$, then an accurate and stable numerical estimate of the exact solution can be achieved.

Tables 1 and 2 show the recovery of the unknowns $x_1^s = 0.662$ cm and k_1 and k_2 given by equation (26), when solving the constraint minimization problem given by equations (24) and (25), when exact and 1% pressure measurements are inputted, respectively. In these tables

Table 1
The retrieval of a piecewise homogeneous thermal conductivity and its discontinuity location for exact data when $N_w \in \{1, 2, 3\}$ and $N_T = 3$

BEM	x_1^s	k_1	k_2
Exact	0.662	4.190	2.095
$\lambda_1 = 1$	0.784	2.928	2.928
$\lambda_2 = 1$	0.595	4.183	2.219
$\lambda_3 = 1$	0.569	4.710	1.997
$\lambda_1 = \lambda_2 = 1$	0.628	4.189	2.099
$\lambda_2 = \lambda_3 = 1$	0.662	4.190	2.095
$\lambda_1 = \lambda_3 = 1$	0.710	3.904	2.094
$\lambda_1 = \lambda_2 = \lambda_3 = 1$	0.668	4.154	2.086

Table 2
The retrieval of a piecewise homogeneous thermal conductivity and its discontinuity location for $p\% = 1\%$ noisy data when $N_w \in \{1, 2, 3\}$ and $N_T = 3$

BEM	x_1^s	k_1	k_2
Exact	0.662	4.190	2.095
$\lambda_1 = 1$	0.765	2.824	2.823
$\lambda_2 = 1$	0.684	4.273	2.084
$\lambda_3 = 1$	0.567	4.710	1.956
$\lambda_1 = \lambda_2 = 1$	0.732	4.103	2.132
$\lambda_2 = \lambda_3 = 1$	0.651	4.329	2.102
$\lambda_1 = \lambda_3 = 1$	0.701	3.828	2.010
$\lambda_1 = \lambda_2 = \lambda_3 = 1$	0.659	4.080	2.004

when the variables λ_i for $i \in \{1, 2, 3\}$, are not specified, they are set up to be zero.

In defining the set of constraints M given by equation (25), the upper and lower physical bounds on the variables were taken to be

$$\underline{bu}^k = (10, 10)\ \mathrm{W\ cm^{-1}\ ^\circ C^{-1}},$$

$$\underline{bu}^s = (1.13)\ \mathrm{cm},$$

$$\underline{bl}^k = (10^{-4}, 10^{-4})\ \mathrm{W\ cm^{-1}\ ^\circ C^{-1}},$$

$$\underline{bl}^s = (0.3)\ \mathrm{cm} \tag{31}$$

and the initial guesses used for initiating the NAG routine E04UCF were given by

$$x_1^s{}^* = \frac{L}{2} = 0.58\ \mathrm{cm},$$

$$k_1^* = 4\ \mathrm{W\ cm^{-1}\ ^\circ C^{-1}},$$

$$k_2^* = 2\ \mathrm{W\ cm^{-1}\ ^\circ C^{-1}}. \tag{32}$$

It should be noted that other initial guesses, such as

$$x_1^s{}^* = \frac{L}{2} = 0.58\ \mathrm{cm},$$

$$k_1^* = 10^{-2}\ \mathrm{W\ cm^{-1}\ ^\circ C^{-1}},$$

$$k_2^* = 10^{-3}\ \mathrm{W\ cm^{-1}\ ^\circ C^{-1}} \tag{33}$$

have also been considered. For all the cases tested in Tables 1 and 2, whether an optimal minimum for the least-squares functional LS, given by equation (24), is attained, or if the current point solution cannot be improved since the differences obtained between successive iterations decrease extremely slowly, the number of iterations needed for such requirements was found to be similar when using either of the initial guesses given by equation (32) or (33). However, in the case $N_w = 1$ the numerical estimates of the exact solution may differ significantly when different initial guesses are adopted and hence show the non-identifiability of the ill-posed problem that one has formulated. For exact data, the numerical results presented in Table 1 show consistency with the conclusions from the sensitivity analysis performed earlier. From Table 1 it can be seen that when $N_w = 1$ and $x_1 = 0$, i.e. $\lambda_1 = 1$, then only $N_T = 3$ time temperature measurements at the boundary $x = 0$ cannot identify any of the unknowns. In addition, the case $\lambda_1 = 1$ shows that $k_1 \approx k_2$ and therefore in such a situation a homogeneous thermal conductivity given by $k \approx 2.928$ W cm^{-1} °C^{-1} may also be considered as a solution for a conductor which initially had a piecewise homogeneous thermal conductivity given by equation (26). When $N_w = 1$ and $x_1 = (x_1^s/2)$, i.e. $\lambda_2 = 1$, then the $N_T = 3$ time temperature measurements at the corresponding interior location can be used to identify only the value of k_1, whilst when $x_1 = (3x_1^s/2)$, i.e. $\lambda_3 = 1$, only the value of k_2, can be identified, see also equation (29). However, as will be shown later, by increasing the number of time measurements N_T to approximately seven or eight and imposing conditions at the single corresponding interior spatial location when $N_w = 1$, then with the heater both on and off it may be possible to produce better estimates of the unknowns. Finally, from Table 1 it can be seen that when $N_w \in \{2, 3\}$, then the $N_T = 3$ time temperature measurements at each of the corresponding space locations produce good estimates of the exact solution, with the best accuracy obtained for $N_w = 2$ interior spatial locations at $x_1 = (x_1^s/2)$ and $x_2 = (3x_1^s/2)$, i.e. in the case $\lambda_2 = \lambda_3 = 1$. Overall, from Table 1 it can be concluded that, for exact data, when the number of time measurements at a sensor location is limited to $N_T = 3 = 2N_s - 1$, then temperature measurements at $N_w = 2$ space locations are needed to be imposed in equation (8) for identifiability and a good retrieval of the unknowns.

The conclusions obtained from Table 1 are confirmed in Table 2 where $p\% = 1\%$ noisy data is inputted in equation (8). This noisy data is generated by perturbing the solution of the direct problem by random Gaussian distributed variables with mean zero and standard deviations σ_{ij}, namely,

$$T(x_i, t'_j)^{(e)} = T(x_i, t'_j) + \varepsilon_{ij}, \varepsilon_{ij} = G05DDF(0, \sigma_{ij}),$$
$$\sigma_{ij} = \frac{p}{100} \times T(x_i, t'_j), \quad i = \overline{1, N_w}, \quad j = \overline{1, N_T} \quad (34)$$

where $G05DDF(m, \sigma)$ is a NAG routine which is designed to generate random Gaussian variables with mean m and standard deviation σ. Table 2 shows good estimates of the exact solution when the inverse problem is identifiable, i.e. when $N_w \geqslant 2$. In such situations, the relative errors between the exact and numerically obtained values are reasonably small and comparable to the amount of noise p%. However, as for exact data in Table 1, it should be noted that the best stable estimate of the solution is obtained in the case $\lambda_2 = \lambda_3 = 1$ corresponding to $N_w = 2$ interior thermocouples located at $x_1 = (x_1^s/2)$ and $x_2 = (3x_1^s/2)$. The amount of redundant noisy data which may have been unwantedly included in the overall number of imposed conditions by increasing to $N_w \times N_T \in \{6, 9\}$, when $N_w \in \{2, 3\}$, is negligible in comparison with the strong information supplied for identifiability and the less severeness of the ill-posed problem.

Next, for both exact and noisy data, we investigate the effect on the numerical inverse solution of increasing the number of time measurements, N_T. It is noted at this stage that for exact data by increasing N_T we expect to improve on the resolution of the numerical solution as more exact information is imposed. However, for noisy data the situation may be different as more noisy redundant data might be unwantedly imposed in equation (8). The time measurements were recorded at equidistant BEM discretisation sampling time nodes, namely,

$$t'_j \in \begin{cases} \tilde{t}_{5j}, & j = \overline{1, N_T}, \quad \text{(if the heater is on)}, \\ \{\tilde{t}_{5j}, \tilde{t}_{N+5j}\}, & j = \overline{1, N_T}, \quad \text{(if the heater is on and off)}. \end{cases} \quad (35)$$

Table 3 shows the recovery of the unknowns x_1^s, k_1 and k_2, when exact and 1% noisy measurements of the temperature at the single location $x_1 = (x_1^s/2)$, i.e. $\lambda_2 = 1$, are recorded at various $N_T \in \{3, 4, 5, 6, 7, 8\}$ times when the heater is on, and on and off. Initially, as predicted by equation (28), it was found that increasing N_T at the single boundary location $x = 0$ did not improve on the non-uniqueness of the inverse solution as previously obtained for the case $\lambda_1 = 1$ in Tables 1 and 2. Therefore, Table 3 includes the results obtained when only interior temperature measurements are considered, i.e. $\lambda_1 = 0$. As expected from equation (29), in Table 3 only the value of k_1 can be retrieved very accurately. However, there is a significant improvement in the prediction of the values of x_1^s and k_2 when the number of time measurements increases from $N_T = 3$ to the interval 4–8. When the heater is on, it can be seen that there is little difference between the results obtained when N_T increases from 4–8 and, therefore, in this case it can be concluded that $N_T = 4$ is the optimal minimal number of time measurements. Better estimates of the exact values can be obtained by recording temperature measurements in time when the heater is both on and off. In this case, it can be seen that this measurement information improves significantly the results with the best accuracy obtained when N_T increases to about seven or eight time measurements. Overall from Table 3 it can be concluded that the measurement at the single location $x_1 = (x_1^s/2)$ can be used successfully for retrieving the unknowns provided that about $N_T \in \{7, 8\}$ time measurements recorded when the heater is both on and off are imposed. Furthermore, the inclusion of noise in this data did not produce any large or oscillatory deviations of the numerical results from their exact values showing that the numerical solution is also stable.

The conclusions obtained from Table 3 are partially validated in Table 4 in which the recovery of the unknowns is shown when the single sensor, recording temperature measurements with or without noise, is located at $x_1 = (3x_i^s/2)$, i.e. $\lambda_3 = 1$, and various numbers of time measurements N_T are investigated. As expected from equation (29), we cannot identify both x_1^s and k_1 even when increasing the number N_T of time measurements at the single sensor location $x = (3x_1^s/2)$. This conclusion is in contrast with that obtained in Table 3, but is somewhat expected as this location $x = (3x_1^s/2)$ is closer to the boundary $x = L$ where a zero temperature condition is prescribed, see equation (4), and the flow is not as developed as at the location $x = (x_1^s/2)$. Also, it can be seen that the estimation of the coefficient k_1 is invariant to the number of time measurements N_T and to whether the heater is on, or on and off. However, consistent with the results of Table 3, it can be seen from Table 4 that the estimation of the coefficients x_1^s and k_2 is improved when temperature measurements in time are recorded when the heater is both on and off in comparison with the case when measurements are taken only when the heater is on.

Overall from Tables 1–4 it can be concluded that for the single discontinuity case investigated in this section, in general, $N_w = 2$ interior space measurements located on each side of the discontinuity need to be imposed for an accurate and stable estimation of the unknowns x_1^s, k_1 and k_2, when the number of time measurements is limited to $N_T = 3$. On the other hand, if the number of sensors N_w is limited to one, then further improvement in retrieving the unknowns may be achieved by increasing the number of time measurements to about $N_T \in \{7, 8\}$ and also by recording temperature data when the heater is both on and off. This improvement is more enhanced when the single sensor is located at the left-hand side of the discontinuity in the region where the thermal-contact boundary condition (5) is applied.

Table 3
The retrieval of a piecewise homogeneous thermal conductivity and its discontinuity location for exact, i.e. $p = 0$, and $p = 1$ noisy data when $\lambda_2 = 1$ and $N_T \in \{3, 4, 5, 6, 7, 8\}$ and the heater is on and on and off

Noise BEM	$p = 0$ x_1^s	$p = 1$ x_1^s	$p = 0$ k_1	$p = 1$ k_1	$p = 0$ k_2	$p = 1$ k_2
Exact	0.662	0.662	4.190	4.190	2.095	2.095
$N_T = 3$ (on)	0.595	0.684	4.191	4.281	2.223	2.012
(on and off)	0.632	0.626	4.190	4.281	2.153	2.136
$N_T = 4$ (on)	0.694	0.687	4.189	4.281	2.020	2.005
(on and off)	0.630	0.624	4.190	4.281	2.156	2.140
$N_T = 5$ (on)	0.696	0.689	4.189	4.281	2.014	2.000
(on and off)	0.630	0.624	4.190	4.281	2.155	2.138
$N_T = 6$ (on)	0.692	0.687	4.189	4.281	2.023	2.005
(on and off)	0.633	0.627	4.190	4.281	2.150	2.133
$N_T = 7$ (on)	0.693	0.696	4.189	4.281	2.022	1.985
(on and off)	0.637	0.632	4.190	4.281	2.142	2.124
$N_T = 8$ (on)	0.699	0.703	4.189	4.281	2.009	1.969
(on and off)	0.642	0.638	4.190	4.281	2.131	2.111

Table 4
The retrieval of a piecewise homogeneous thermal conductivity and its discontinuity location for exact, i.e. $p = 0$, and $p = 1$ noisy data when $\lambda_3 = 1$ and $N_T \in \{3, 4, 5, 6, 7, 8\}$ and the heater is on and on and off

Noise BEM	$p = 0$ x_1^s	$p = 1$ x_1^s	$p = 0$ k_1	$p = 1$ k_1	$p = 0$ k_2	$p = 1$ k_2
Exact	0.662	0.662	4.190	4.190	2.095	2.095
$N_T = 3$ (on)	0.570	0.576	4.814	4.769	2.033	2.009
(on and off)	0.576	0.577	4.719	4.718	2.044	2.022
$N_T = 4$ (on)	0.563	0.565	4.718	4.718	1.973	1.944
(on and off)	0.574	0.576	4.719	4.718	2.040	2.018
$N_T = 5$ (on)	0.565	0.561	4.718	4.718	1.981	1.934
(on and off)	0.574	0.575	4.718	4.718	2.040	2.018
$N_T = 6$ (on)	0.556	0.557	4.718	4.718	1.948	1.923
(on and off)	0.571	0.577	4.718	4.718	2.035	2.024
$N_T = 7$ (on)	0.553	0.553	4.718	4.718	1.942	1.909
(on and off)	0.578	0.583	4.718	4.718	2.053	2.041
$N_T = 8$ (on)	0.547	0.548	4.718	4.718	1.923	1.896
(on and off)	0.582	0.588	4.718	4.718	2.064	2.054

Furthermore, some preliminary studies on heat conductors presenting multiple faults, i.e. $N_s > 2$, indicate that, in general, $N_w \in \{(N_s - 1), N_s\}$ interior temperature measurements recorded at $N_T \geqslant (2N_s - 1)$ instants need to be inverted in order to ensure an identifiable and good retrieval of the piecewise homogeneous thermal conductivity and its discontinuity points for a sample subjected to the heat flow test investigated in this study.

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